Problem 1

(9 pts each)Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = cos\left(\pi + \frac{ln\, n}{n}\right)$

$\lim_{n\to+\infty} \frac{\ln n}{n} = 0$ Basic limit

$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \cos\left(\pi + \frac{\ln n}{n}\right) = \cos\left(\pi + \lim_{n\to+\infty} \frac{\ln n}{n}\right)$

$= \cos(\pi + 0) = \cos(\pi) = -1$

So a_n converge to -1 as n goes to ∞

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(b) $b_n = 5^{\frac{(-1)^n}{n}}$

For $(-1)^n = 1$

$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} 5^{\frac{1}{n}} = 5^0 = 1$

For $(-1)^n = -1$

$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} 5^{-\frac{1}{n}} = 5^0 = 1 = \lim_{n\to+\infty} b_n$ for $(-1)^n = 1$

So $\lim_{n\to+\infty} b_n = 1$

7 8!

Continuity

$\lim_{n\to+\infty} \frac{\alpha^n}{n!} = 0$ for $\alpha > 1$

~~Basic limit~~

(c) $c_n = \frac{2^n + n!}{2^n - n!}$

$\lim_{n\to+\infty} \frac{2^n + n!}{2^n - n!} = \lim_{n\to+\infty} \frac{\cancel{2^n}\left(1 + \frac{n!}{2^n}\right)}{\cancel{2^n}\left(1 - \frac{n!}{2^n}\right)} = \lim_{n\to+\infty} \frac{1 + \frac{n!}{2^n}}{1 - \frac{n!}{2^n}}$

$\lim_{n\to+\infty} \frac{2^n + n!}{2^n - n!} = \lim_{n\to+\infty} \frac{\cancel{n!}\left(\frac{2^n}{n!} + 1\right)}{\cancel{n!}\left(\frac{2^n}{n!} - 1\right)} = \lim_{n\to+\infty} \frac{\frac{2^n}{n!} + 1}{\frac{2^n}{n!} - 1} = \frac{0+1}{0-1} = -1$

c_n converges to -1

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Problem 2

(9 pts each) Which of the following series converge, and which diverge? Find the sum of the series when possible.

(a) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{4^n}-\frac{2^{n+1}}{5^n}\right)$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}-\frac{2^{n+1}}{5^n} = \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n - \sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n - 2\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n$$

$\left(-\frac{1}{4}\right)^n$ and $\left(\frac{2}{5}\right)^n$ are geometric series of reason

$-1 < r < 1$

So

$$\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n - 2\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{1}{1-\left(-\frac{1}{4}\right)} - 2\times\frac{1}{1-\frac{2}{5}}$$

$$= \frac{4}{5} - 2\times\frac{5}{3}$$

$$= \frac{4}{5} - \frac{10}{3} = \cancel{\frac{4-20}{5}}\ \frac{12-50}{15} = -\frac{38}{15}$$

$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n} - \frac{2^{n+1}}{5^n}$ converges to $-\frac{38}{15}$

$$\lim_{n\to+\infty}\left(1+\frac{c}{n}\right)^{n}=e^{c}$$

Basic limit

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+4}{n+3}\right)^{n}$

$\displaystyle\sum_{n=1}^{\infty}$ nth term test

$$\cancel{\lim_{n\to+\infty}\left(\frac{n+4}{n+3}\right)^{n}=\lim_{n\to+\infty}\left(\frac{n+3+1}{n+3}\right)^{n}}$$

$$\cancel{=\lim_{n\to+\infty}\left(1+\frac{1}{n+3}\right)^{n}}$$

$$\lim_{n\to+\infty}\left(\frac{n+4}{n+3}\right)^{n}=\lim_{n\to+\infty}\left(\frac{n(1+\frac{4}{n})}{n(1+\frac{3}{n})}\right)^{n}=\lim_{n\to+\infty}\frac{(1+\frac{4}{n})^{n}}{(1+\frac{3}{n})^{n}}$$

$$=\frac{e^{4}}{e^{3}}=e\neq 0$$

So $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+4}{n+3}\right)^{n}$ diverges

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^{n}}{n\sqrt{n^{2}+1}}\right)$

$$a_n=\frac{(-1)^{n}}{n\sqrt{n^2+1}}$$

Let's look at $\displaystyle\sum_{n=1}|a_n|$

$$\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2+1}}$$

$$n^2+1>n^2$$

$$\sqrt{n^2+1}>\sqrt{n^2}$$

$$\frac{1}{\sqrt{n^2+1}}<\frac{1}{\sqrt{n^2}}$$

$$\frac{1}{n\sqrt{n^2+1}}<\frac{1}{n\sqrt{n^2}}$$

So $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2+1}}<\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n^2}}=\sum_{n=1}^{\infty}\frac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges being a p series with $p>1$

So By DCT $\displaystyle\sum_{n=1}^{\infty}|a_n|$ converges

So By ACT that means that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n\sqrt{n^2+1}}$ converges

Problem 3

(a)(12 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+2^n}\right)x^n .$$

(Remember to check the endpoints.)

$\lim_{n\to+\infty} \frac{n}{\alpha^n} = 0$ for $\alpha > 1$

Basic limit

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+2^n}\right) x^n \qquad a_n = \left(\frac{1}{n+2^n}\right)\cdot x^n$$

Ratio test

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{1}{n+1+2^{n+1}}\cdot|x|^{n+1} \times \frac{n+2^n}{|x|^n}$$

$$= \lim_{n\to+\infty} \frac{n+2^n}{n+1+2^{n+1}}\cdot|x|$$

$$= \lim_{n\to+\infty} \frac{\cancel{2^{n+1}}\left(\frac{n}{2^n}+\frac{1}{2}\right)}{\cancel{2^{n+1}}\left(\frac{n}{2^n}+\frac{1}{2^n}+1\right)}\times|x|$$

$$= \frac{0+\frac{1}{2}}{0+0+1}\cdot|x|$$

$$= \frac{1}{2}|x| = \rho$$

$\sum_{n=1}^{\infty}\left(\frac{1}{n+2^n}\right)x^n$ converges if and Only if

$$\frac{1}{2}|x| < 1$$

$$-1 < \frac{1}{2}x < 1$$

$$\times 2 \quad -2 < x < 2 \quad \times 2$$

So for $x \in]-2;2[$ $\sum_{n=0}^{\infty} a_n$ Converges. Let's check the endpoints

For $x = +2$

$$\sum_{n=1}^{\infty} \frac{1}{n+2^n}\cdot 2^n = \sum_{n=1}^{\infty} \frac{2^n}{n+2^n} \text{ nth term test}$$

$$\lim_{n\to+\infty} \frac{2^n}{n+2^n} = \lim_{n\to+\infty} \frac{2^n}{2^n\left(\frac{n}{2^n}+1\right)} = \lim_{n\to+\infty} \frac{1}{\frac{n}{2^n}+1} = \frac{1}{0+1} = 1 \neq 0$$

So for $x = 2$ $\sum_{n=0}^{\infty} a_n$ diverges by nth term test

Please check the backpage for the remainder of Problem 3

Grade

For $x=-2$

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+2^n} = \frac{1}{1+2} - \frac{2}{1+4} +$$

~~ratio~~

~~For $(-2)^n = 2^n$ $\sum_{n=0}^{\infty} a_n$ diverges (it's the same one as x=2)~~

~~For $(-2)^n$~~ $= -2^n$

$\sum_{n=0}^{\infty}$ nth term test

$$\lim_{n\to+\infty} \frac{(-2)^n}{n+2^n} = \lim_{n\to+\infty} \frac{2^n\left(\frac{-2}{2}\right)^n}{2^n\left(\frac{n}{2^n}+1\right)} = \lim_{n\to+\infty} \frac{(-1)^n}{\frac{n}{2^n}+1}$$

For $(-1)^n = 1$

$$\lim_{n\to+\infty} \frac{1}{\frac{n}{2^n}+1} = \frac{1}{0+1} = 1$$

For $(-1)^n = -1$

limit doesn't exist

$$\lim_{n\to+\infty} \frac{-1}{\frac{n}{2^n}+1} = \frac{-1}{0+1} = -1 \neq 1 \neq 0$$

So $\sum_{n=0}^{\infty} \frac{(-2)^n}{n+2^n}$ diverges

So the interval of convergence of $\sum_{n=0}^{\infty}\left(\frac{1}{n+2^n}\right)x^n$ is

$x \in]-2;2[$

(b)(6 pts) For $x = -1$, use the alternating series estimation theorem (ASET) to approximate the series in part **(a)** with an error of magnitude less than $\frac{1}{36}$. Decide if your answer is an over-estimate or an under-estimate. (Make sure to justify why the conditions for ASET are satisfied.)

For $x=1$ $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2^n}$

S_N is the partial sum of N terms of $\sum_{n=0} \frac{(-1)^n}{n+2^n}$

L is the limit of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2^n}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2^n} = \cancel{1} - \frac{1}{1+2} + \frac{1}{2+4} - \frac{1}{3+8} + \dots$$

$= u_1 + u_2 - u_3 + u_4 + \dots$ So the series is alternating

and also $u_1 > u_1 > u_3 > u_4 \dots$ because $n+2^n$ increases with n so $\frac{1}{n+2^n}$ decreases

$$\lim_{n\to\infty} \frac{(-1)^n}{n+2^n} = \lim_{n\to\infty} \frac{2^n\left(\frac{-1}{2}\right)^n}{2^n\left(\frac{n}{2^n}+1\right)} = \lim_{n\to\infty} \frac{\left(\frac{-1}{2}\right)^n}{\frac{n}{2^n}+1} = \frac{0}{0+1} = 0$$ So AST is verified

So by using ASET ~~we know~~ that

~~$|S_N - L| < u_{n+1}$~~

So when $u_{n+1} < \frac{1}{36}$ $|S_N - L| < \frac{1}{36}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2^n} = \cancel{1} - \frac{1}{1+2} + \frac{1}{2+4} - \frac{1}{3+8} + \frac{1}{4+16} - \frac{1}{5+32} + \frac{1}{6+64} - \frac{1}{7+128} + \dots$$

$$= \cancel{1} - \frac{1}{3} + \frac{1}{6} - \frac{1}{11} + \frac{1}{20} - \frac{1}{37} + \frac{1}{70} - \frac{1}{135} + \dots$$

Check the backpage for the rest of problem 3(b)

(c)(6 pts) Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2^n} \leq \frac{x}{2-x} \qquad \text{for all } 0 \leq x < 2.$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2^n} < \sum_{n=1}^{\infty} \frac{x^n}{2^n}$$

$\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$ is a geometric series for $0 \leq x \leq 2$

so $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n = \frac{x/2}{1-\left(\frac{x}{2}\right)} = \frac{2 \times \frac{x}{2}}{2-x} = \frac{x}{2-x}$

So for $0 \leq x \leq 2$

$$\sum_{n=0}^{\infty} \frac{x^n}{n+2^n} \leq \frac{x}{2-x}$$

Grade

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2^n} = \overbrace{-\frac{1}{3} + \frac{1}{6} - \frac{1}{11} + \frac{1}{20}}^{\text{Estimation}} \overbrace{- \frac{1}{37} + \frac{1}{70} - \frac{1}{35} \dots}^{\text{error}}$$

$$|u| = +\frac{1}{37} < \frac{1}{36}$$

So the first four term of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2^n}$ are sufficient to get an approximation of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2^n}$

So the approximation with an error of magnitude less than $\frac{1}{36}$ of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2^n}$ is

$$S_4 = -\frac{1}{3} + \frac{1}{6} - \frac{1}{11} + \frac{1}{20} \cancel{\epsilon - \frac{1}{6}}$$

~~n this is an underestimate because~~

$$\cancel{\sum_{n=1}^{\infty} = S_4 - \frac{1}{37} + \frac{1}{70} - \dots}$$

$$= \cancel{S_4}$$

that is an overestimate because the last term is + $\frac{1}{20}$

Problem 4

(10 pts) Use the fact that $(ln(1+x))' = \frac{1}{1+x}$ to find a power series expansion for the function $f(x) = ln(1+x)$ about the center $a = 0$. Remember to mention the values of x for which your steps are justified.

$\frac{1}{1+x} = \frac{1}{1-(-x)}$ of the for $\frac{1}{1-r}$

So for $-1 < -x < 1 \Leftrightarrow -1 < x < 1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$

$\int \frac{1}{1+x} dx = \int \sum_{n=0}^{\infty} (-x)^n dx$

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$\cancel{\int \frac{1}{1+x} dx = \sum_{n=0}^{\infty} \int (-x)^n}$

~~ln(1~~ Let's take $u = 1+x$

$\int \frac{u'}{u} du = \int \cancel{x} - x + x^2 - x^3 + x^4 - \ldots \, dx$

$\ln u + c = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}$

$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} + C$ for $-1 < x < 1$

Exchange x for 0 to find c

$\ln(1+0) = \ln 1 = 0$ so

$\sum_{n=0}^{\infty} \frac{(0)^{n+1}}{n+1} + c = 0$

$0 + 0 + 0 + \ldots + c = 0$

$c = 0$

So $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}$ for $-1 < x < 1$

of center $a = 0$

Problem 5

(a)(6 pts) State and prove the nth term test for series.

The nth term test states that if $\lim_{n\to+\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

~~and if $\lim_{n\to+\infty} a_n = 0$ $\sum_{n=0}^{\infty} a_n$ converges~~

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Let's suppose $\sum_{n=1}^{\infty} a_n$ converges and let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$ and ~~L be~~ and the limit ~~of~~ of the series be L

$\cancel{\lim_{N\to+\infty} S_N} = L$ When $N \to +\infty$ S_N and S_{N+1} tend to L

$S_{N+1} - S_N = (a_{n+1} + a_n + a_{n-1} + \dots + a_2 + a_1) - (a_n + a_{n-1} + a_{n-2} + \dots + a_2 + a_1)$

$= a_{n+1}$

$\lim_{n\to+\infty} S_{N-1} - S_N = \lim_{n\to+\infty} a_n$ 4

$\lim_{n\to+\infty} a_n = L - L = 0$

So if $\sum_{n=1}^{\infty} a_n$ converges $\lim_{n\to+\infty} a_n$ must be equal to 0

(b)(6 pts) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Suppose that

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$. Without using the ratio test use the result of part **(a)**

to prove that $\sum_{n=1}^{\infty} a_n$ diverges.

~~$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 2$~~

~~$\lim_{n\to+\infty} a_{n+1} = 2 \lim_{n\to+\infty} a_n$~~

~~$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty}$~~

$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 2$

~~$\lim_{n\to+\infty} e^{\ln\left(\frac{a_{n+1}}{a_n}\right)} = e^{\ln 2}$~~

~~$\lim_{n\to+\infty} e^{\ln(a_{n+1}) - \ln a_n}$~~

$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = 2$

$\lim_{n\to+\infty} \ln \left(\frac{a_{n+1}}{a_n}\right) = \ln 2$ $\ln x$ is a continuous function for all $x > 0$

$\lim_{n\to+\infty} \ln(a_{n+1}) - \ln(a_n) = \ln 2$

But if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} a_{n+1}$

So $\lim_{n\to+\infty} \ln(a_{n+1}) - \ln(a_n) \neq \ln 2$

So $\sum_{n=0}^{\infty} a_n$ diverges